

# Dimension data, and Local versus Global Conjugacy in Reductive Groups

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To Prof. Robert Langlands

## 1 Introduction

Let  $G$  be a reductive algebraic group and  $H$  be a connected semisimple subgroup of  $G$ . The dimension data consists of the collection  $\{\dim V^H\}$  with  $V^H$  denoting the space of points in  $V$  fixed by  $H$ , and where  $(\rho, V)$  runs through all representations of  $G$ . One may ask if the dimension data determine  $H$  up to conjugacy or at least isomorphism. In other words, If  $H$  and  $H'$  have the same dimension data, are they isomorphic? If so, are they conjugate in  $G$ ?

For the first question, when  $G = GL(n, \mathbb{C})$ , M. Larsen and R. Pink ([L-P90]) gave an affirmative answer. i.e., if  $H$  and  $H'$  are two semisimple subgroups of  $GL(n, \mathbb{C})$  and have the same dimension data, then  $H$  and  $H'$  must be isomorphic. Moreover, as each algebraic group embeds into  $GL(n)$ , the answer is still “yes” for the first question when we replace  $G$  by an arbitrary complex reductive algebraic group.

For the second question, M. Larsen and R. Pink also gave an answer in the same paper for a special case. They proved when  $G = GL(n, \mathbb{C})$ , and  $H, H'$  irreducible (i.e., not contained in any proper Levi subgroup of  $G$ ), having the same dimension data forces them to be conjugate in  $G$ . This fact is used in our article.

Our first result is that, the answer is still “yes” if  $G = SO(2n+1)$ ,  $Sp(2n)$  and  $O(N)$  with certain conditions. More precisely, we have the following:

**Theorem A.** *Let  $\rho$  and  $\rho'$  be two homomorphisms from a connected semisimple complex group  $H$  into  $G = O(N)$ ,  $SO(N)$ , or  $Sp(N)$  ( $N = 2n$ ), which is naturally embedded into  $GL(N)$ . And assume that*

- (i)  $\rho$  and  $\rho'$  are irreducible in  $GL(N)$ ;
- (ii)  $\rho$  and  $\rho'$  possess the same dimension data.

*Then  $\rho(H)$  and  $\rho'(H)$  differ by an automorphism of  $G$ . Moreover, if  $G = O(N)$ ,  $Sp(2n)$  or  $SO(2n+1)$ , this automorphism is inner so that  $\rho$  and  $\rho'$  are globally conjugate in image.*

However, this is *not* the case in general. In this article we give a family of pairs  $(G, H)$  with  $G = SO(2N, \mathbb{C})$ , such that the dimension data does not determine  $H$  up to conjugacy. In addition, our examples are *connected, complex semisimple groups*, so that they give rise to connected analogues to known examples due to Larsen and Blasius ([Larsen94], [Larsen96], [Blasius94]) where  $H$  is *discrete and finite*.

Our second result, which is the main one of the paper, is the following:

**Theorem B.** *Let  $H$  be a simple Lie group over  $\mathbb{C}$  with its Lie algebra being one of the following types:*

$$A_{4n}(n \geq 1), B_{2n}(n \geq 2), C_{2n}(n \geq 2), E_6, E_8, F_4, G_2$$

*and  $G = SO(2N)$  where  $2N = \dim \text{Lie}(H)$ . Then there exist two embeddings  $i$  and  $i'$  of  $H$  into  $G$ , such that their images  $i(H)$  and  $i'(H)$  are not conjugate in  $G$ , but they possess the same dimension data. In fact,  $i$  and  $i'$  are locally conjugate, but not globally conjugate in image.*

Here “locally conjugate” means that  $i(h)$  and  $i'(h)$  are conjugate in  $G$  for each semisimple  $h$ ; “globally conjugate in image” means that  $i(H)$  and  $i'(H)$  are conjugate in  $G$  (cf. Section 2).

The simplest example comes when  $H = SO(5)$ ,  $G = SO(10)$ ,  $i = \text{Ad}$ , the adjoint representation, or  $\Lambda^2$ , the exterior square, and  $i' = \tau \circ i$  where  $\tau$  is some automorphism on  $SO(10)$  which is not inner. In fact, we consider the situation with the following conditions:  $H = \text{Int}(\mathfrak{g})$  where  $\mathfrak{g}$  is a simple Lie algebra of even rank,  $i$  is the adjoint representation  $\text{Ad}$  of  $H$  whose image in  $GL(\mathfrak{g})$  is contained in  $G = SO(\mathfrak{g}, \kappa) \cong SO(2N)$  where  $\kappa$  is the Killing form of  $\mathfrak{g}$ , and  $i' = \tau \circ i$  where  $\tau$  is some automorphism on  $G = SO(\mathfrak{g}, \kappa)$ . The list given in Theorem B exhausts all possible cases in this situation.

We will prove Theorem A in Section 2 and Theorem B in Section 3 and 4. One may observe that our construction and proof are still available if  $\mathbb{C}$  is replaced by any algebraic closed field of characteristic 0 such as  $\bar{\mathbb{Q}}$  or  $\bar{\mathbb{Q}}_l$ . In fact, a potential application is in comparing for two continuous homomorphisms  $\rho_l$  and  $\rho'_l$  from  $\mathcal{G}_k$  to  $G(\bar{\mathbb{Q}}_l)$  for  $\mathcal{G}_k$  the absolute Galois group for a number field  $k$  such that the Zariski closure  $H_l$ ,  $H'_l$  of  $\text{Im}(\rho_l)$  and  $\text{Im}(\rho'_l)$  respectively have same dimension data. By Tchebotarev, the Frobenius classes  $\text{Frob}_v$ , for  $v$  unramified in  $\rho_l$  and  $\rho'_l$ , generate the Galois images topologically, and local conjugacy implies that for any algebraic representation  $r: G \rightarrow GL_N$ , the  $L$ -factors  $L_v(s, r \circ \rho_l)$  and  $L_v(s, r \circ \rho'_l)$  are the same.

We show in Section 6 that if certain instances of Langlands’ Principle of functoriality are known, then our examples give rise to cusp forms  $\pi$  on certain  $SO(2n)/F$ ,  $F$  a number field, which appear with multiplicity bigger than one. The interest in this is that the conjectural Langlands group  $H(\pi)$  in such an example will be a connected subgroup of the dual group  $\hat{G}$ . Earlier instances

of such failures of multiplicity one like for  $SL(n)(n \geq 3)$  ([Blasius94]) because of the local vs global conjugacy issues, are associated to  $H(\pi)$ 's which are disconnected, even finite. For  $SL(2)$ , one knows multiplicity one ([Ra2000]), and for  $n \geq 3$ , E.M. Lapid ([Lapid99]) showed that under a Tanakian formalism, the multiplicities are bounded for each  $n$ . However, this is not the case for  $G_2$  ([Gan-Gurevich-Jiang])<sup>1</sup>.

**Theorem C.** *Let  $2N = \dim \text{Lie}(H)$ , where  $H$  is a simple Lie group of one of the following type*

$$A_{4n}(n \geq 1), B_{2n}(n \geq 1), C_{2n}(n \geq 1), E_6, E_8, F_4, G_2$$

*Assume the following conditions:*

(1) *The Langlands functoriality from  $H$  to  $SO(2N)$  which arises from the adjoint representation  $i$  of  $H$  holds. Also, The Langlands functoriality for  $i' = \tau \circ i$  also holds, where  $\tau$  is an outer automorphism of  $SO(2N)$ .*

(2) *The Arthur's conjecture, including the Arthur's multiplicity formula for  $SO(2N)$  holds.*

(3) *There is a cuspidal automorphic representation  $\pi_0$  of  $H$ , which is associated to a global  $l$ -adic Galois representation  $\rho_l$  with  $\text{Im}(\rho_l)$  Zariski dense in the dual group  $\hat{H}(\mathbb{Q}_l)$ .*

*Then the multiplicity one fails for  $G = SO(2N)$ . Moreover, there are two nearly equivalent cuspidal representations  $\pi$  and  $\pi'$ , not of finite Galois type, which cannot be distinguished by their incomplete  $L$ -functions, i.e., for a finite set  $S$  of places,*

$$L^S(s, \pi, r) = L^S(s, \pi', r)$$

*for any algebraic representation  $r$  of  $\hat{G}$ ; yet they don't occur in the same constituent and hence give multiplicity  $> 1$ .*

For a proof, see Section 6.

This result was suggested by Langland's paper "Beyond Endoscopy" ([Langlands2002]), and this Theorem gives nontrivial evidence for his prediction.

In a sequel we plan to investigate the instances of functoriality when  $F$  is a function field over a finite field, by appealing to Lafforgue's work as well as the recent forward and backward lifting results of generic forms from the classical groups to  $GL(n)$ . The transfer in the forward direction was done for number

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<sup>1</sup>For  $G_2$  in [Gan-Gurevich-Jiang], the reason of failure of multiplicity one is different from that for  $SL(n)$ . Indeed, in  $G_2$ , it is expected that local conjugacy implies global conjugacy. The reason for the failure of multiplicity one in  $G_2$  is because the Arthur multiplicity formula gives answers which are  $> 1$ . For classical groups, Arthur's formula always gives 1 or 0; yet multiplicity one fails at the times because local-global conjugacy does not hold.

fields by Cogdell, Kim, Piatetski-Shapiro and Shahidi in [CoKPSS] (and the papers to follow), and very recently, progress on the analogue for function fields over finite fields was done by L.A. Lomeli in his Purdue thesis. The failure of multiplicity one for suitable  $SO(2n)$  comes from the outer automorphism  $\tau$  of  $SO(2n)$  which arises from a conjugation in  $O(2n)$ . The fact that cusp forms on  $SO(2n)$  which are moved by the outer automorphism will appear with multiplicity is one of the key observations of this work and it points to a need to fine tune Arthur's original multiplicity conjectures. It raises the following question: Can one use the twisted trace formula relative to  $\tau$  and isolate the cusp forms with multiplicity one. We hope to investigate this in another sequel.

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## 2 Local Conjugacy and Global Conjugacy

Let  $H$  and  $H'$  be two semisimple subgroups of  $G$ . We say that they are **locally conjugate** if there is an isomorphism  $\tau$  from  $H$  to  $H'$  such that  $\tau(h)$  and  $h$  are conjugate in  $G$  for each semisimple  $h \in H$ ;  $H$  and  $H'$  are **globally conjugate** if they differ by a conjugation, i.e., an inner automorphism of  $G$ . Moreover, let  $i$  and  $i'$  be two embeddings of  $H$  into  $G$ . We say that  $i$  and  $i'$  are **locally conjugate** if  $i(h)$  and  $i'(h)$  are conjugate in  $G$  for each semisimple  $h \in H$ ;  $i$  and  $i'$  are **globally conjugate in image** if  $i(H)$  and  $i'(H)$  are globally conjugate in  $G$ , i.e.,  $i'(H) = c(i(H))$  for some inner automorphism  $c$  of  $G$ ; They are **globally conjugate** if  $i' = c \circ i$  for some inner automorphism  $c$  of  $G$ .

*Remark:* If  $i, i': H \hookrightarrow G$  are globally conjugate in image, then there exists an automorphism  $\tau$  of  $H$ , and an element  $t \in G$  such that, for each  $h \in H$ ,  $i'(\tau(h)) = t^{-1}i(h)t$ .

**Proposition 2.1.** *If  $H$  and  $H'$  are locally conjugate in  $G$ , then they have the same dimension data.*

*Proof.* Let  $i$  be the identity map on  $H$  and  $i'$  be the isomorphism from  $H$  to  $H'$  such that  $i'(h)$  and  $i(h)$  are conjugate for each  $h \in K(H)$  where  $K(H)$  is a maximal compact subgroup of  $H$ . This is possible since all elements on a maximal compact subgroup are semisimple. Then for each representation  $(\sigma, V)$  of  $G$ , we have

$$\begin{aligned} \dim V^H &= \text{mult}(1, \sigma \circ i) \\ &= \frac{1}{\text{Vol}(K(H))} \int_{K(H)} \text{Tr} \sigma(h) d\mu \end{aligned}$$

and similarly,

$$\begin{aligned} \dim V^{H'} &= \text{mult}(1, \sigma \circ i') \\ &= \frac{1}{\text{Vol}(K(H))} \int_{K(H)} \text{Tr}(\sigma \circ i')(h) d\mu \end{aligned}$$

Thus to prove that  $H$  and  $H'$  have the same dimension data, it suffices to prove the following claim:  $\text{Tr} \sigma(h) = \text{Tr}(\sigma \circ i')(h)$  for any  $h$  in  $K(H)$ . As  $H$  and  $H'$  are locally conjugate,  $i(h) = h$  and  $i'(h)$  are conjugate in  $G$  for each  $h$  in  $K(H)$ . So  $i'(h) = \beta^{-1} h \beta$  for some  $\beta$  in  $G$ . So,  $(\sigma \circ i')(h) = \sigma(\beta)^{-1} \sigma(h) \sigma(\beta)$ . Hence the claim and the theorem.  $\square$

Remark: This theorem still applies for  $G$  over  $\bar{\mathbb{Q}}_l$ , the algebraic closure of  $\mathbb{Q}_l$ . The reason is that, for any algebraic subgroup  $H$ , and any algebraic representation  $(\sigma, V_{K'})$  of  $G$ ,

$$\dim_{\bar{\mathbb{Q}}_l} V_{\bar{\mathbb{Q}}_l}^H = \dim_{K'} V_{K'}^H = \dim_{\mathbb{C}} V_{\mathbb{C}}^H$$

where  $K'$  is the number field where  $i$  and  $\sigma$  split.

Next, we prove Theorem A.

Let  $\rho$  be an irreducible selfdual representation of a complex semisimple group  $H$  into  $GL(V)$ . Then  $\rho$  is either orthogonal or symplectic, and in either case, the image of  $\rho$  must fix some symmetric or alternating nondegenerate bilinear form  $\omega$  on  $V$ . In fact, such form must be unique up to scalar.

**Lemma 2.2.** *Let  $\rho$  be an irreducible selfdual finite dimensional representation of a connected semisimple group  $H$ . Then There is a unique nondegenerate bilinear form  $\omega$  up to a scalar that  $H$  preserves. i.e.,  $\rho(H)(\omega) = \omega$ .*

*Proof.* Note that each nondegenerate bilinear form, either symmetric or alternating, up to a scalar, that  $H$  fixes (or fixes up to a scalar factor) corresponds a trivial (or a one dimensional) constituent of  $\rho \otimes \rho$ . Hence, it suffices

to prove that,  $\rho \otimes \rho$  possesses exactly one one-dimensional constituent. This is the case since following: First,  $H$  is semisimple and selfdual. and thus  $\rho$  is equivalent to  $\tilde{\rho}$  which is the contragradient of  $\rho$ . Moreover, the multiplicity of one occurred in  $\rho \otimes \tilde{\rho}$  is exactly the square sum of the multiplicities of the irreducible constituents of  $\rho \otimes \tilde{\rho}$ , which is 1 if  $\rho$  is irreducible.

□

**Lemma 2.3.** *Let  $\rho$  and  $\rho'$  be two irreducible representations of a connected complex semisimple group  $H$  into  $GL(V) = GL(N, \mathbb{C})$ . Assume that*

(i)  $\rho(H)$  and  $\rho'(H)$  fixes the same nondegenerate bilinear form  $\omega$ , either symmetric or alternating;

(ii)  $\rho$  and  $\rho'$  are globally conjugate in  $GL(V)$ .

*Then they are globally conjugate in  $O(V, \omega)$ , i.e., there is a  $t \in GL(V)$  such that,  $\rho'(g) = t^{-1}\rho(g)t$  for each  $g$  in  $H$ , and  $t$  fixes  $\omega$ .*

*Proof.* Let  $t$  be an element in  $GL(V)$  such that  $\rho'(g) = t^{-1}\rho(g)t$  for each  $g$  in  $H$ . Then we claim that some scalar multiple of  $t$ , namely,  $t_1 = ct$ , fixes  $\omega$ . Granting this, we have  $t_1 \in O(V, \omega)$ , and  $\rho'(g) = t_1^{-1}\rho(g)t_1$  for each  $g$  in  $H$ . Hence the lemma.

Now we prove the claim. Check that,  $H$  fixes  $t\omega$ . In fact, for each  $h \in H$ ,

$$\rho(h)(t\omega) = (\rho(h)t)(\omega) = (t\rho'(h))(\omega) = t(\rho'(h)(\omega)) = t\omega$$

So by Lemma 2.2, together with the assumption that  $\rho$  is irreducible,  $t\omega = c_1\omega$  for some nonzero  $c_1$ . Thus the claim.

□

### ***Proof of Theorem A.***

First we treat the case when  $G = O(N)$  or  $Sp(N)$ . Then  $G = O(N, \omega)$  for some nondegenerate bilinear form  $\omega$ , either symmetric or symplectic on  $V$  where  $V$  is the  $N$ -dimensional space where  $G \subset GL(N)$  acts.

Note that the dimension data of  $(H, G)$  are a sub-collection of the dimension data of  $(H, GL(N))$ . Then (i), (ii), plus the Larsen–Pink imply that  $\rho$  and  $\rho'$  are globally conjugate in image in  $GL(N)$ . Hence there is some  $\varphi \in \text{Aut}(H)$  such that  $\rho \circ \varphi$  and  $\rho'$  are globally conjugate in  $GL(N)$ . Note that  $\rho \circ \varphi$  is also irreducible. Hence by Lemma 2.3,  $\rho \circ \varphi$  and  $\rho'$  are globally conjugate in  $O(N)$ . Hence the theorem.

For  $G = SO(2n+1)$ , we also have the theorem as  $O(2n+1) = \{\pm I\}SO(2n+1)$ .

□

*Remark:* One expects that, if  $G$  does not allow outer automorphisms, then the second question raised in the introduction of this article is affirmative, with only the assumption that,  $H$  or  $H'$  are not contained in any proper parabolic or Levi subgroups of  $G$ .

### 3 Proof of Theorem B, Part I

Let  $\mathfrak{g}$  be a simple Lie algebra with the Killing form  $\kappa$  defined as  $\kappa(X, Y) = \text{Tr}(ad(X)ad(Y))$ . It is clear that  $\kappa$  is nondegenerate by the semisimplicity. Let  $H$  be the adjoint semisimple Lie group with Lie algebra  $\mathfrak{g}$ . Here *adjoint* means  $Z(H)$  is trivial. Such  $H$  exists and is unique by the Lie theory. In fact,  $H \simeq \text{Int}(\mathfrak{g})$ . Hence, the adjoint representation of  $H$  gives rise to an irreducible representation of  $H$  in the underlying space of  $\mathfrak{g}$ :

$$i : H \hookrightarrow Gl(\mathfrak{g})$$

**Lemma 3.1.**  *$H$  preserves the Killing form. So that, the image of  $H$  lies in  $SO(\mathfrak{g}, \kappa)$ .*

Here for each linear space  $V$  and a nondegenerate symmetric bilinear form  $\omega$ ,  $SO(V, \omega)$  consists of linear transforms that preserves  $\omega$ . It is clear that  $SO(V, \omega)$  is isomorphic to  $SO(\dim V)$ .

*Proof.* For each  $X, Y$  and  $Z$  in  $\mathfrak{g}$ ,  $c \in \text{Int}(\mathfrak{g})$ ,

$$\begin{aligned} \text{ad}(c(X))\text{ad}(c(Y))(Z) &= [c(X), [c(Y), Z]] \\ &= [c(X), c([Y, c^{-1}(Z)])] = c([X, [Y, c^{-1}(Z)])] \\ &= c(\text{ad}(X)\text{ad}(Y)c^{-1}(Z)) \end{aligned}$$

Hence  $\text{ad}(c(X))\text{ad}(c(Y)) = c \circ \text{ad}(X)\text{ad}(Y) \circ c^{-1}$  is conjugate to  $\text{ad}(X)\text{ad}(Y)$ . So they must have the same trace.

Furthermore,  $H$  is connected and so is its image. Therefore, the image must lie in  $SO(\mathfrak{g}, \kappa)$ , which is the identity connected component of  $O(\mathfrak{g}, \kappa)$ . Done.  $\square$

Now we make several assumptions to help to explain our examples.

Assumption (A): the rank of  $\mathfrak{g}$  is even.

So by Lemma 3.1 under Assumption (A),  $i$  is orthogonal, and its image lies in  $SO(\mathfrak{g}, \kappa)$  which is isomorphic to  $SO(2N, \mathbb{C})$  where  $2N$  is the dimension of  $\mathfrak{g}$ .

Let  $\tau$  be any “odd” element in  $O(\mathfrak{g}, \kappa)$ , i.e.,  $\tau$  preserves  $\kappa$  but does not lie in the identity component  $SO(\mathfrak{g}, \kappa)$ . Also, let  $i' = \text{Conj}(\tau) \circ i$ , i.e.,  $i'(x) = \tau i(x) \tau^{-1}$ .

Let  $\Phi$  be the root system of  $H$  and  $\Delta$  be a base. We first assume the following:

Assumption (B):  $\text{Aut}(\Phi) = \text{Inn}(\Phi)$ .

Assumption (B) is equivalent to that, the only isometry of the Dynkin diagram is the identity map. Under such assumption, the Lie automorphism of  $H$  must be a conjugacy by some element in  $G$ . The simple Lie algebras satisfying Assumption (B) are  $A_1$ ,  $B_n$  ( $n \geq 2$ ),  $C_n$  ( $n \geq 2$ ),  $E_7$ ,  $E_8$ ,  $F_4$  and  $G_2$ .

In next section, we will deal with the case when  $\Phi$  allows a outer automorphism.

**Proposition 3.2.** *With the Assumption (A) and (B),  $i$  and  $i'$  are locally conjugate, but not globally conjugate in image. The only cases that apply are  $\mathfrak{g} = B_{2n}$ ,  $C_{2n}$ ,  $E_8$ ,  $F_4$  and  $G_2$ .*

*Remark:* So the simplest example should be  $\mathfrak{g} = B_2 = C_2$ . In this example,  $H$  is  $SO(5)$ , and the adjoint representation of  $H$  gives rise to the exterior square from  $SO(5)$  to  $SO(10)$ .

We need two lemmas for the proof.

**Lemma 3.3.** *Let  $A$  and  $B$  are two diagonalizable matrices in  $SO(2N)$ , and suppose that they are conjugate in  $O(2N)$ , and 1 occurs as an eigenvalue for either  $A$  or  $B$ . Then  $A$  and  $B$  are conjugate in  $SO(2N)$ .*

*Proof.* First, we claim that  $A$  commutes with an “odd” element in  $O(2N)$ , i.e., an element in  $O(2N)$  but not  $SO(2N)$ .

We choose an appropriate coordinate system so that the invariant quadratic form on  $O(2N, \mathbb{C})$  is  $\omega = \sum x_i y_i$ . So, with respect to a basis  $\{e_1, f_1, e_2, f_2, \dots, e_n, f_n\}$ , the matrix representing  $\omega$  is  $\text{Diag}(P, P, \dots, P)$  where  $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . By conjugation in  $SO(2N)$  and without loss of generality, we may also assume  $A$  is diagonal, i.e., of the form  $\text{diag}(a_1, a_1^{-1}, a_2, a_2^{-1}, \dots, a_n, a_n^{-1})$ . So if say  $a_1 = 1$ , then the centralizer of  $A$  contains  $O(2) \times Id$  where  $O(2)$  is the orthogonal group on  $\langle e_1, f_1 \rangle$  and  $Id$  is the identity map on the orthogonal complement  $\langle e_2, f_2, \dots, e_n, f_n \rangle$ . Hence the claim.

Now assume that  $B = C^{-1}AC$ . If  $C \in SO(2N)$  we are done. Otherwise,  $C$  is an “odd” element. Hence,

$$B = C^{-1}AC = C^{-1}Q^{-1}AQC = (QC)^{-1}A(QC)$$

for some “odd” element  $Q$  which commutes with  $A$ . Thus  $QC \in SO(2N)$ , and hence  $A$  and  $B$  must be also conjugate in  $SO(2n)$ .  $\square$

**Lemma 3.4.** *Let  $\rho : H \rightarrow GL(n, \mathbb{C})$  be an irreducible orthogonal representation. Then the centralizer of  $H$  in  $O(n, \mathbb{C})$  is  $\{\pm I\}$ .*



*Proof.* As  $\rho$  is irreducible,  $C_{GL(n, \mathbb{C})}(\rho(H)) = C^*$  from the Schur's Lemma. Then  $C_{O(n, \mathbb{C})}(\rho(H)) = C_{GL(n, \mathbb{C})} \cap O(n, \mathbb{C}) = C^* \cap O(n, \mathbb{C}) = \{\pm I\}$ .  $\square$

***Proof of Proposition 3.2.***

*Step 1.*  $i$  and  $i'$  are locally conjugate.

For each  $h_0$  semisimple,  $h = i(h_0)$  and  $h' = i'(h_0) = \tau h \tau^{-1}$ , where  $\tau \in O(\mathfrak{g}, \kappa) \simeq O(2N)$ . Hence  $h$  and  $h'$  are conjugate in  $O(\mathfrak{g}, \kappa)$ . Moreover, they are semisimple also as  $i$  and  $i'$ , and hence each of them must be contained in some complex torus, so that  $h$  fixes some Cartan subalgebra in  $\mathfrak{g}$ . This shows that  $h$  has eigenvalue 1 in  $SO(\mathfrak{g}, \kappa)$ . Furthermore the multiplicity of 1 is at least the dimension of this Cartan Subalgebra, namely, the rank of  $\mathfrak{g}$  which is at least 2 from Assumption (A) which says that the rank of  $\mathfrak{g}$  is even. Thus Lemma 3.3 applies, and  $h$  and  $h'$  must be conjugate in  $SO(\mathfrak{g}, \kappa)$ . By the arbitrary choice of  $h_0$ ,  $i$  and  $i'$  are locally conjugate.

*Step 2.*  $i$  and  $i'$  are not globally conjugate in image.

Now we need to use Assumption (B). Assume the contrary, that  $i$  and  $i'$  are globally conjugate in image, and moreover, that the conjugation by  $\beta \in SO(\mathfrak{g}, \kappa)$  sends  $i'(H)$  to  $i(H)$ , i.e.,  $i(H) = \beta i'(H) \beta^{-1}$ . Thus the conjugation by  $\beta \tau$  restricted to  $i(H)$  is an automorphism on  $i(H) \simeq H$ . From Assumption (B), all Lie automorphisms on  $i(H) \simeq H$  must be inner. Then this automorphism is in fact the conjugation by some element  $\gamma$  in  $i(H) \subset SO(\mathfrak{g}, \kappa)$ . Hence  $\gamma^{-1} \beta \tau$  must lie in the centralizer of  $i(H)$  in  $O(\mathfrak{g}, \kappa) \simeq O(2N)$  since the conjugation by  $\gamma^{-1} \beta \tau$  fixes  $i(H)$  at all. As  $\mathfrak{g}$  is simple, then the adjoint representation of  $H$  in  $\mathfrak{g}$  must be irreducible. Hence by Lemma 3.4,  $\gamma^{-1} \beta \tau$  must be  $\pm I$  which is obviously an element in  $SO(\mathfrak{g}, \kappa)$ . However,  $\tau$  is “odd”, and  $\beta$  and  $\gamma$  are “even” as they lie in  $SO(\mathfrak{g}, \kappa)$ . Then  $\gamma^{-1} \beta \tau$  must be “odd”. This gives a contradiction.

Hence  $i$  and  $i'$  are not globally conjugate in image.

Then the first part of this theorem follows. The second part is clear as the only connected Dynkin diagrams of even rank without any outer automorphisms are  $B_{2n}$ ,  $C_{2n}$ ,  $E_8$ ,  $F_4$  and  $G_2$ .  $\square$

*Remark:* When Proposition 3.2 applies,  $i(H)$  and  $i'(H)$  have the same dimension data. But they are not necessarily conjugate in  $G = SO(\mathfrak{g}, \kappa)$ .

## 4 Proof of Theorem B, Part II

Consider the adjoint representations of simple adjoint groups. What happens if the root system  $\Phi$  admits a nontrivial outer automorphism? For example, consider the type  $A_n$ ,  $D_n$  ( $n \geq 3$ ),  $E_6$ . We will prove, if  $\mathfrak{g}$  is  $A_{4n}$  or  $E_6$ , we still

have the same conclusion, i.e,  $i$  and  $i'$  are locally conjugate but not globally conjugate in image.

In fact, it is enough for us to replace Assumption (B) by the following weaker assumption.

Assumption (C): All elements of  $\text{Aut}(H)$  are “even” in the adjoint representation.

First, we claim that, given an adjoint representation  $i : H = \text{Int}(\mathfrak{g}) \rightarrow GL(\mathfrak{g})$  where  $\mathfrak{g}$  is a simple Lie algebra of even degree, if  $\tau_1$  is an automorphism of  $H$ , which induces an automorphism  $\tau'_1$  of  $\mathfrak{g}$ , then  $\tau_1$  is the restriction of  $\text{Conj}(\tau'_1)$  on  $GL(\mathfrak{g})$  to  $H$ , i.e.,

$$\text{Ad}(\tau_1(h))(X) = \tau'_1 \circ \text{Ad}(h) \circ \tau'^{-1}_1(X)$$

for each  $h \in H$  and  $X \in \mathfrak{g}$ .

**Definition** Assume that the rank of  $\mathfrak{g}$  is even.  $\tau_1$  is said to be “even” if  $\tau'_1$ , viewed as a linear transformation, has determinant 1, and “odd” if  $\tau'_1$  has determinant  $-1$ .

*Remark.* If  $\tau_1$  is inner, then it must be “even” since  $\tau'_1$  lies in the image of  $H$  in  $GL(\mathfrak{g})$ , hence in  $SO(\mathfrak{g}, \kappa)$  as  $H$  is connected, thus having determinant 1.

**Lemma 4.1.** *For each  $\tau_1 \in \text{Aut}(H)$ ,  $\tau'_1$  lies in  $O(\mathfrak{g}, \kappa)$ . Hence  $\tau_1$  is “even” if and only if  $\tau'_1$  lies in  $SO(\mathfrak{g}, \kappa)$ . In particular, all the inner automorphisms of  $H$  are “even”.*

*Proof.* The idea is almost the same as in the proof of Lemma 3.1. For each  $X, Y$  and  $Z$  in  $\mathfrak{g}$ , we have

$$\begin{aligned} \text{ad}(\tau'_1(X))\text{ad}(\tau'_1(Y))(Z) &= [\tau'_1(X), [\tau'_1(Y), Z]] \\ &= [\tau'_1(X), \tau'_1([Y, \tau'^{-1}_1(Z)])] = \tau'_1([X, [Y, \tau'^{-1}_1(Z)])] \\ &= \tau'_1(\text{ad}(X)\text{ad}(Y) \tau'^{-1}_1(Z)) \end{aligned}$$

Here we used the fact that  $\tau'_1$  preserves the Lie brackets as it comes from  $\text{Aut}(H)$  and hence a Lie algebra automorphism.

Hence  $\text{ad}(\tau'_1(X))\text{ad}(\tau'_1(Y))$  and  $\text{ad}(X)\text{ad}(Y)$  have the same trace. Hence  $\tau'_1$  preserves the Killing form of  $\mathfrak{g}$ . The rest assertions are now clear.  $\square$

Recall that  $i$  denotes the adjoint representation of  $H$ , and  $i' = \text{Conj}(\tau) \circ i$ . Here  $H$  is an adjoint group with simple Lie algebra  $\mathfrak{g}$ .

**Proposition 4.2.** *With the Assumption (A) and (C),  $i$  and  $i'$  are locally conjugate, but not globally conjugate in image.*

**Proof of Proposition 4.2.**

Recall that in the proof of Proposition 3.2, Assumption (B) is used only in Step 2. Hence the local conjugacy of  $i$  and  $i'$  is clear when we replace Assumption (B) with (C). Thus it suffices to redo Step 2.

*Step 2'.*  $i$  and  $i'$  are not globally conjugate in image.

Now we need to use Assumption (C) instead of Assumption (B). Recall that  $i'(H) = \tau^{-1}i(H)\tau$ . Now again assume the contrary, i.e.,  $i$  and  $i'$  are globally conjugate in image, and again that  $i(H) = \beta i'(H)\beta^{-1}$  for  $\beta \in SO(\mathfrak{g}, \kappa)$ . Then the conjugation by  $\beta\tau$  restricted to  $i(H) \sim H$  is an automorphism on  $H$ . From Assumption (C), and Lemma 4.1, it induces the conjugation by  $\gamma \in SO(\mathfrak{g}, \kappa)$ . Hence  $\gamma^{-1}\beta\tau$  must lie in the centralizer of  $i(H)$  in  $O(\mathfrak{g}, \kappa) \simeq O(2N)$  since the conjugation by  $\gamma^{-1}\beta\tau$  fixes  $i(H)$  at all. As  $\mathfrak{g}$  is simple, then the adjoint representation of  $H$  in  $\mathfrak{g}$  must be irreducible. Hence by Lemma 3.4,  $\gamma^{-1}\beta\tau$  must be  $\pm I$  which is obviously an element in  $SO(\mathfrak{g}, \kappa)$ . However,  $\tau$  is “odd”, and  $\beta$  and  $\gamma$  are “even” as they lie in  $SO(\mathfrak{g}, \kappa)$ . Then  $\gamma^{-1}\beta\tau$  must be “odd”. This gives a contradiction also.

Hence  $i$  and  $i'$  are not globally conjugate in image. □

Now we want to refine further and prove that Assumption (C) can in fact be replaced by the following equivalent assumption.

Assumption (C'): All automorphisms of  $\Phi$  that preserves a base is an even permutation on the base. i.e., All automorphisms of the Dynkin diagram are even permutations of vertices.

In fact, since each automorphism of  $H$  is a product of an inner automorphism which is “even” by Lemma 4.1, and an automorphism that preserves a maximal torus  $T$  and a base  $\Delta$  of the root system, we may focus on those  $\tau_1$  that preserves  $T$  and  $\Delta$ . Denote  $\mathfrak{t}$  be the Cartan subalgebra corresponding to  $T$ . Then  $\tau'_1$ , induced from  $\tau_1$  as a Lie algebra automorphism on  $\mathfrak{g}$ , preserves  $\mathfrak{t}$ . For each positive root  $\beta$ , let

$$V(\beta) = \mathfrak{g}_\beta + \mathfrak{g}_{-\beta}$$

where  $\mathfrak{g}_{\pm\beta}$  is the root space of  $\pm\beta$  respectively. Then  $\mathfrak{g}$  is a direct sum of  $V = \oplus V(\beta)$  and  $\mathfrak{t}$ .

Hence, as  $\tau_1$  permutes positive roots,  $\tau_1$  permutes  $V(\beta)$  for all positive roots  $\beta$ . In fact,  $\tau'_1$  sends  $\mathfrak{g}_\beta$  to  $\mathfrak{g}_{\tau_1(\beta)}$ , and  $\mathfrak{g}_{-\beta}$  to  $\mathfrak{g}_{\tau_1(-\beta)}$ . Thus  $\tau'_1$  restricted to  $V$  is “even”, i.e, it has determinant 1, and the sign of  $\tau'_1$  is hence the same as the sign of  $\tau'_1$  restricted to  $\mathfrak{t}$ .

Moreover, as  $\tau'_1$  sends  $H_\beta$  to  $H_{\tau_1(\beta)}$  for each  $\beta \in \Delta$ , where  $H_\beta$  is the co-root associated to  $\beta$  in  $\mathfrak{t}$ , the sign of the restriction of  $\tau'_1$  to  $\mathfrak{t}$  is the same as the sign of permutation on  $\Delta$  by  $\tau_1$ , and it is the same as the sign of the automorphism of Dynkin graph induced by  $\tau_1$ . Thus, we have proved the following:

**Lemma 4.3.** *Assumption (C') is equivalent to Assumption (C).*

**Corollary 4.4.** *The only additional cases where Proposition 4.2 applies are  $\mathfrak{g} = A_{4n}$  and  $E_6$ .*

*Proof.* Let  $\mathfrak{g}$  be a simple Lie algebra of even rank, and assuming that  $\mathfrak{g}$  admits an outer automorphism. Then  $\mathfrak{g}$  is of type  $A_{2n}$ ,  $D_{2n}$  or  $E_6$ . Suppose in addition that Assumption (C) holds, i.e., all automorphisms of the Dynkin graph are even permutations. For  $A_{2n}$ , the Dynkin graph allows only one outer automorphism, which is a product of  $n$  transpositions on the vertices. Thus Assumption (C') will rule out  $A_{2n}$  when  $n$  is odd, and keep  $A_{4n}$ .  $D_{2n}$  is ruled out also as the Dynkin diagram obviously admits an automorphism which swaps two nodes and keeps others.  $E_6$  remains as the only outer automorphism of the Dynkin graph is a product of two transpositions on the vertices.

Hence, by Lemma 4.3, these are the only cases Proposition 4.2 applies.  $\square$

So Theorem B follows by combining of Proposition 3.2 and Proposition 4.2. When Assumption (C), or equivalently Assumption (C') fails,  $\Phi$  and hence  $\mathfrak{g}$  will allow an “odd” automorphism, so that any two subgroups of  $SO(2N)$  conjugate in  $O(2N)$  must be also conjugate in  $SO(2N)$ . Hence  $i$  and  $i'$  are also globally conjugate. This explains why the list given in Theorem B exhausts all possibilities in this situation.

## 5 Preliminaries on Langlands Functoriality and Multiplicity One

In this part, let us recall some preliminary facts on Langlands Functoriality and the multiplicity one. The experts can skip this section and go directly to Section 6.

Fixing a ground field  $F$ , a local or global field, let  $G$  be an algebraic group,  $\mathcal{A}(G)$  the set of automorphic representations of  $G(\mathbb{A}_F)$  where  $\mathbb{A}_F$  denotes the adele ring of  $F$ , and  $\mathcal{A}_0(F)$  denote the set of cuspidal automorphic representations of  $G(\mathbb{A}_F)$ . Also,  ${}^L G = \hat{G} \rtimes \mathcal{W}_F$  denotes the Langlands dual group of  $G$ , and  ${}^L G^\circ = \hat{G}$  is the connected component of  ${}^L G$ . For details, see [Bo] and [Cogdell2003-1]. For the known instances of the local Langlands, see

also [Langlands89] (archimedean case), [LRS93] (non-archimedean case, positive characteristic), [HaT2001] (Non-archimedean case, characteristic 0), and [Jiang-So2003], [Jiang-So2004] (for  $SO_{2n+1}$ ).

Now, we state the global Langlands conjecture, which is not proved in general. Now let  $F$  be a global field. We need conjectured  $\mathcal{L}_F$ , the Langlands group of  $F$  taking the role of local Weil or Weil–Deligne groups (for the description, see [LL79], [Arthur02]). When  $\pi$  is of Galois type, then the global parametrization  $\phi$  can be taken as a Galois representation. Moreover, when we use  $l$ -adic representations instead of  $\mathbb{C}$ -representations, the global Weil group  $\mathcal{W}_F$  is believed to serve for this role.

If the local Langlands for each local group  $G_v = G(F_v)$  is assumed, then attaching to  $\pi = \bigotimes \pi_v$  we give a collection of local admissible representations  $\{\phi_v\}$  of local parameters  $\phi_v = \phi_{\pi_v}: \mathcal{W}_{F_v} \rightarrow {}^L G(F_v)$ . Let  $\iota_v$  be the natural embedding of  ${}^L G_v = \hat{G} \rtimes \mathcal{W}_{F_v}$  into  ${}^L G$ . Then, one gets a collection  $\{\iota_v \circ \phi_v\}$  of local parameters  $\iota_v \circ \phi_v: \mathcal{W}_{F_v} \rightarrow {}^L G$ .

For any finite set of places  $S$  of  $F$ , we can define a family of global (incomplete)  $L$ -functions of  $\pi$  as

$$L^S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r_v) = \prod_{v \notin S} L(s, r \circ \iota_v \circ \phi_v)$$

when  $r$  runs through all representations from  ${}^L G \rightarrow GL_N(\mathbb{C})$  and  $r_v = r \circ \iota_v$ .

**The Principle of Functoriality.** (cf. [Langlands2002])

*If  $k$  is a local or global field,  $H$  and  $G$  be connected reductive  $k$ -groups with  $G$  quasisplit, then to each  $L$ -homomorphism  $u: {}^L H \rightarrow {}^L G$  there is associated a transfer/lifting of admissible or automorphic representations of  $H$  to admissible or automorphic representations of  $G$ .*

Let  $\pi = \bigotimes \pi_v$  and  $\pi' = \bigotimes \pi'_v$  be two automorphic representations of  $G$ , and  $\{\rho_v\}$  and  $\{\rho'_v\}$  are the families of local Weil representations to  ${}^L G$  associate to  $\pi$  and  $\pi'$  respectively. We say that,  $\pi$  and  $\pi'$  are “locally conjugate”, if for almost all  $v$ ,  $\pi_v \cong \pi'_v$ , i.e.,  $\pi$  and  $\pi'$  are nearly equivalent, or equivalently,  $\rho_v$  and  $\rho'_v$  are “locally conjugate” in  ${}^L G$  for almost all  $v$ . Since for almost all  $v$  the local  $L$ -factor  $\rho_v$  is unramified, and uniquely determined by the image of the Frobenius elements, then  $\pi$  and  $\pi'$  cannot be distinguished by their incomplete  $L$ -functions if they are locally conjugate.

Let  $\pi = \bigotimes \pi_v$  be an automorphic representation of  $G(\mathbb{A}_F)$ .  $m(\pi)$ , the multiplicity of  $\pi$  is defined as the multiplicity of  $\pi$  occurred in  $L^2_{\text{cusp}}(G(F) \backslash G(\mathbb{A}_F))$ , and it is positive if and only if  $\pi$  is cuspidal. Fixing a conjectural global parameter  $\rho$  of  $\pi$ ,  $m(\mathcal{L}(\pi)) = m(\mathcal{L}(\pi, \rho))$ . The multiplicity of an (stable)  $L$ -packet of  $\pi$  (which is singleton if  $G = GL(n)$ ), is defined as the

multiplicity of any  $\pi'$  in  $\mathcal{L}(\pi)$  which is the set of cuspidal automorphic representations  $\pi'$  associated to the same global parameter  $\rho$  as  $\pi$ , and it is the number which Arthur's multiplicity formula for the packet  $\mathcal{L}(\rho)$  produces.  $m_{\text{global}}(\pi)$  is defined as the sum of  $m(\mathcal{L}(\pi'))$  where  $\mathcal{L}(\pi')$  runs through all different  $L$ -packets  $\mathcal{L}(\pi')$  where  $\pi$  and  $\pi'$  are nearly equivalent, i.e.,  $\pi'_v \cong \pi_v$  for almost  $v$ . This is the multiplicity that occurs in the Arthur's multiplicity formula. We will be mainly concerned with parameters which are *tempered*, and for an explication of the Arthur's multiplicity formula in this case we refer the readers to Lapid's paper [Lap99]. We say that  $G$  satisfies the multiplicity one if  $m(\pi) = m_{\text{global}}(\pi) = 1$  for all cuspidal  $\pi$ . We know the multiplicity one for  $GL(n)$  ([JPSS83], [JS81], [JS90]) and  $SL(2)$  ([Ra2003]).

## Some Heuristics

Let us consider some heuristics. Let  $F$  be a global field with the conjectured Langlands group  $\mathcal{L}_F$  ([Arthur02], [Langlands2002], [Cl], [Ra-La]), and for each place  $v$ , let  $\mathcal{L}_{F_v}$  be the corresponding local group, which can be taken to be the Weil group  $\mathcal{W}_{F_v}$  for  $v$  archimedean, and  $\mathcal{W}_{F_v} \times SL_2(\mathbb{C})$  for  $v$  non-archimedean. There should be natural morphism  $j_v: \mathcal{L}_{F_v} \rightarrow \mathcal{L}_F$  such that any global parameter, i.e., a homomorphism  $\phi: \mathcal{L}_F \rightarrow {}^L G$  will induce local parameters  $\phi_v = \phi \circ j_v$ .

Let  $i: {}^L H \rightarrow {}^L G$  be an  $L$ -homomorphism such that  ${}^L H^\circ(\mathbb{C})$  and  ${}^L G^\circ(\mathbb{C})$  are connected, and the image of  ${}^L H^\circ = \hat{H}$  is not contained in any proper parabolic subgroup of  ${}^L G$ . Let  $\pi$  be a cuspidal automorphic representation of  $G(\mathbb{A}_F)$  whose parameter  $\phi_\pi: \mathcal{L}_F \rightarrow {}^L G$  satisfies the following:

- (a)  $\phi_\pi$  factors through  ${}^L H$ .
- (b)  $\text{Im}(\phi_\pi)$  is dense in  ${}^L H$ .

This implies in particular that, for each place  $v$ , the local parameter  $\phi_{\pi_v}: \mathcal{L}_{F_v} \rightarrow {}^L G$  factors through  ${}^L H$ . Note that the Langlands group  $\tilde{H}(\pi)$  is essentially the Zariski closure of the image of  ${}^L H$ .

When  $G = SO(2N)$ ,  ${}^L G^\circ = \hat{G} = SO(2N, \mathbb{C})$ , and we can produce examples  $\pi$  with multiplicities  $> 1$ , if we assume functoriality for  $i: {}^L H \rightarrow {}^L G$ . The main ideal is that when we have two  $L$ -parameters  $\phi$  and  $\phi'$  which are locally conjugate but not globally conjugate in image, they will produce such an example.

To avoid the problem of the existence of  $\mathcal{L}_F$  with desired properties, we will restrict our attention to those  $\pi$  whose parameters are tempered, and moreover naturally representations of the absolute Galois (or Weil) group. This is reasonable because our ultimate aim is to get nontrivial examples which can be analyzed concretely.

## 6 Potential Failure of Multiplicity One for Cusp Forms on $SO(2N)$

Let  $G$  be a reductive algebraic group and  $F$  an number field. Again, for each field  $k$ , denotes the absolute Galois group as  $\mathcal{G}_k$ . Let  $H$  be a semisimple algebraic group such that  $\hat{H}$  embeds into  $\hat{G}$ . Let  $i$  and  $i'$  be two algebraic, injective homomorphisms from  $\hat{H}$  into  $\hat{G}$  such that they are locally conjugate but not globally conjugate in image. We also denote by  $i$  and  $i'$  the induced  $L$ -homomorphisms from  ${}^L H$  to  ${}^L G$ .

We assume that:

(1) The Langlands Functoriality for  $i, i'$  holds, i.e., the functorial transfer of automorphic forms for  $H(\mathbb{A}_F)$  to  $G(\mathbb{A}_F)$  exists corresponding to the  $L$ -homomorphisms  $i, i': {}^L H \rightarrow {}^L G$ , at least for global parameters  $\phi$  attached to Galois representations.

(2) The Arthur's conjecture, including the Arthur's multiplicity formula for  $G$  holds. i.e., the (global) multiplicity of each cuspidal automorphic representation is exactly the same as given in the Arthur's multiplicity formula. More precisely, to each  $L$ -parameter  $\rho$  of  $G = SO(2N)$ , we associate a global  $L$ -packets  $\rho$  and a subrepresentation  $L^2(\rho)$  of  $L^2_{\text{cusp}}$  which is a direct sum of representations of  $\rho$  with multiplicities described by Arthur's multiplicity formula. Moreover, if two parameters  $\rho$  and  $\rho'$  are not conjugate in image, then  $L^2(\rho)$  and  $L^2(\rho')$  must be orthogonal to each other.

(3) There are two cuspidal automorphic representation  $\pi$  and  $\pi'$  of  $G(\mathbb{A}_F)$ , furnished by (1) from  $\pi_0$  of  $H(\mathbb{A}_F)$  with respective parameters  $\phi$  and  $\phi'$ , such that the image of  $\phi$  is dense in  $i(\hat{H})$ .

Let  $\phi_0$  be the parameter of  $\pi_0$ , and then we have  $\phi = i \circ \phi_0$  and  $\phi' = i' \circ \phi_0$ .

First,  $\pi$  and  $\pi'$  are nearly equivalent. In fact, as  $i$  and  $i'$  are locally conjugate, so are  $\phi = \phi_\pi = i \circ \phi_0$  and  $\phi' = \phi_{\pi'} = i' \circ \phi_0$ . Moreover,  $L^S(s, \pi, r) = L^S(s, \pi', r)$  for some finite set of places of  $F$ .

Moreover,  $\pi$  and  $\pi'$  cannot occur in the same constituent as their parameters  $\phi$  and  $\phi'$  are not globally conjugate in image. In fact, as the Zariski closures of  $\phi$  and  $\phi'$  are  $i(\hat{H})$  and  $i'(\hat{H})$  respectively, they are not globally conjugate in  $\hat{G}$ , and hence  $L^2(\phi)$  and  $L^2(\phi')$  are orthogonal to each other in  $L^2_{\text{cusp}}(G(F) \backslash G(\mathbb{A}))$ .

So the global multiplicities of  $\pi$  is at least 2, and this gives rise to an example to fail the multiplicity one for  $G$ .

Before we formulate a theorem in Galois version. we will start from a global  $l$ -adic homomorphism  $\phi_0 : \mathcal{G}_F \rightarrow \hat{H}(\bar{\mathbb{Q}}_l) \hookrightarrow {}^L H(\bar{\mathbb{Q}}_l)$  where  $\hat{H}$  is the dual group of  $H$ , and is always viewed as an algebraic subgroup of  ${}^L H$  with  $\hat{H}(\mathbb{C}) = {}^L H^\circ(\mathbb{C})$ .

Put  $\phi = i \circ \phi_0$  and  $\phi' = i' \circ \phi_0$ . Given  $\phi_0$  and any place  $v$ , we have a natural homomorphism from  $\mathcal{G}_{F_v}$  to  $\mathcal{G}_F$ , and by the restricting  $\phi$  defines  $\phi_v : \mathcal{G}_{F_v} \rightarrow {}^L G(\mathbb{Q}_l)$ , and similarly  $\phi'_v : \mathcal{G}_{F_v} \rightarrow {}^L G(\mathbb{Q}_l)$ .

The local parameter  $\phi_v : \mathcal{G}_{F_v} \rightarrow {}^L G(\mathbb{Q}_l)$  defines a homomorphism, again denoted  $\phi_v$  by abuse of notation, from  $\mathcal{W}'_{F_v}$  to  ${}^L G(\mathbb{C})$  (See Tate's article in Colvallis [Tate]). Hence  $\phi_v$  should defines by functoriality to an irreducible admissible representation  $\pi_v$  of  $G(F_v)$ . This correspondence is known when  $\phi_v$  is unramified. Same thing works for  $\phi'_v$ .

**Theorem 6.1.** *Let  $F$  be a global field and  $l$  a prime not equal to the characteristic of  $F$ . Let  $H$  and  $G$  be two algebraic reductive groups,  $i$  and  $i'$  two algebraic injective morphism (or with finite kernel) from  $\hat{H}(\mathbb{Q}_l)$  to  $\hat{G}(\mathbb{Q}_l)$ , and  $\rho_1$  an  $l$ -adic Galois homomorphism from  $\mathcal{G}_F$  to  $\hat{H}(\mathbb{Q}_l)$ , with all the images are semisimple. Assume that,*

- (1)  *$i$  and  $i'$  are locally conjugate but not globally conjugate in image.*
- (2) *The image of  $\rho_1$  is Zariski dense in  $\hat{H}$ .*

*Let  $\phi = i \circ \rho_1$  and  $\phi' = i' \circ \rho_1$ . Then  $\phi$  and  $\phi'$  are locally conjugate but not globally conjugate in image.*

*Moreover, if  $\phi$  and  $\phi'$  are modular, i.e., if they are associated to  $\pi$  and  $\pi'$ , then  $\pi$  and  $\pi'$  are nearly equivalent, and moreover give rise to multiplicity  $\geq 2$  in the space of cusp forms on  $G(\mathbb{A}_F)$ .*

The proof is similar to the discussion above.

*Proof:*

First,  $\phi$  and  $\phi'$  are locally conjugate. In fact, as  $i$  and  $i'$  are locally conjugate, and all the images  $\rho_1(g)$  are semisimple, then  $\phi = i \circ \rho_1$  and  $\phi' = i' \circ \rho_1$  are also locally conjugate.

Moreover,  $\phi$  and  $\phi'$  are not globally conjugate in image. In fact, the Zariski closures of  $\phi$  and  $\phi'$  are  $i(\hat{H}(\mathbb{Q}_l))$  and  $i'(\hat{H}(\mathbb{Q}_l))$  respectively, and they are not globally conjugate in image in  $\hat{G}$ .

Finally, if  $\phi$  and  $\phi'$  are modular, and are associated to cuspidal automorphic representations  $\pi$  and  $\pi'$  of  $G(\mathbb{A}_F)$  respectively, then  $\pi$  and  $\pi'$  are locally conjugate and hence nearly equivalent. However, they give rise to multiplicity  $\geq 2$  in the space of cusp forms on  $G(\mathbb{A}_F)$  as the images of  $\phi$  and  $\phi'$  are not conjugate.

□

In particular, if  $(\hat{H}, \hat{G})$  comes from our list from Theorem B, i.e,  $H$  is the simply connected algebraic Lie group such that  $\hat{H}$  is one of the simple adjoint Lie group with Lie algebra  $A_{4n}$  ( $n \geq 1$ ),  $B_{2n}$  ( $n \geq 2$ ),  $C_{2n}$  ( $n \geq 2$ ),  $E_6$ ,  $E_8$ ,



$F_4$  and  $G_2$ ,  $i$  is the adjoint representation and  $i' = C \circ i$  where  $C$  is some outer automorphism of  $G = SO(\mathfrak{g}, \kappa) = SO(2N)$ , which is also the conjugation by some  $g_0 \in O(2N) - SO(2N)$ . Applying Theorem B,  $i$  and  $i'$  are locally conjugate but not globally conjugate. Thus Theorem 6.1 and hence Theorem C follow. Note that, if we assume the global Langlands for  $G = SO(2N)$  here, or equivalently, first, assume the global Langlands for  $GL(2N)$ , then assume the Langlands functoriality for the descent from  $SO(2N)$  to  $GL(2N)$ , then Theorem C will also apply automatically.

## 7 An example: Case $B_2$

Now we come to the case when  $\mathfrak{g} = B_2 = C_2$  where  $\mathfrak{g} = \mathfrak{so}(5) = \mathfrak{sp}(4)$ ,  $H = Sp(4)$  with  $\hat{H} = SO(5)$ . As  $\dim(\mathfrak{g}) = 10$ , the adjoint representation  $i$  is also equivalent to the exterior square  $\Lambda^2$ . Then Theorem 6.1 applies. So the assumptions we need in this case for the theorem are:

(I) The functoriality from  $Sp(4)$  to  $SO(10)$  holds for the exterior square or the adjoint representation  $i$  from  ${}^L Sp(4) = SO(5) \rightarrow {}^L SO(10) = SO(10)$ .

(II) The functoriality from  $Sp(4)$  to  $SO(10)$  holds also for  $i' = \tau \circ i$  for some outer Lie automorphism  $\tau$  of  $SO(10)$ .

(II) There is an  $l$ -adic Galois homomorphism  $\rho_1: \mathcal{G}_F \rightarrow SO(5, \bar{\mathbb{Q}}_l)$  whose image is dense.

As  $SO(5) \cong PSp(4) = Sp(4)/\{\pm 1\} = GSp(4)/Z$ , so (II) can be replaced by:

(II') there is a 4-dimensional  $l$ -adic Galois representation  $\rho_1$  such that, the Zariski closure of its image contains  $Sp(4)$ .

Let's quote the following lemma which is an easy result in the Lie representation theory.

**Lemma 7.1.** *Let  $k$  be an algebraically closed field, and  $\rho$  be a 4-dimensional irreducible continuous representation of a connected semisimple algebraic group  $H(k)$  over  $k$  with finite kernel. Then up to equivalence, the pair  $(H, \rho)$  is one of the following:*

- (1)  $H = SL(4)$ , and  $\rho$  is the standard representation.
- (2)  $H = SL(2)$ , and  $\rho = \text{sym}^3$ .
- (3)  $H = Sp(4)$ , and  $\rho$  is the standard representation.
- (4)  $H = SL(2) \times SL(2)$ , and  $\rho = \rho_0 \otimes \rho_0$  where  $\rho_0$  is the standard representation of  $SL(2)$ .
- (5)  $H = SO(4)$ , and  $\rho$  is the standard representation.

Moreover, when  $\rho$  is symplectic, only (2) and (3) are possible, and when  $\rho$  is orthogonal, only (4) and (5) are possible.

□

**Proposition 7.2.** *Let  $\rho_1$  be an irreducible continuous 4-dimensional representation of  $\mathcal{G}_F$  over  $\bar{\mathbb{Q}}_l$ , and assume the following:*

- (a) *The Zariski closure of  $\mathrm{Im}(\rho_1)$  is a reductive algebraic group.*
- (b)  *$\rho_1$  is (essentially) self dual, and is of  $GS(4)$  type;*
- (c)  *$\rho_1$  is not twist equivalent to  $\mathrm{sym}^3(\rho_2)$  for any continuous representation  $\rho_2$  of  $\mathcal{G}_F$  over  $\bar{\mathbb{Q}}_l$ .*

*Then the Zariski closure of  $\mathrm{Im}(\rho_1)$  contains  $Sp(4)$ .*

*Proof:* This can be deduced from the results of [Ra2003]. Let us give the argument for completeness.

Let  $H$  be the Zariski closure of  $\mathrm{Im}(\rho_1)$ , and  $\rho$  be the embedding of  $H$  into  $GL(4)$ . Moreover, denote  $H' = H^{ss}$  be a semisimple part of  $H$ . Note that  $H'(\bar{\mathbb{Q}}_l)$  is unique up to conjugacy.

Then  $\rho$  restricted to  $H' = H^{ss}$  is also irreducible. Hence, from Lemma 7.1,  $H'(\bar{\mathbb{Q}}_l)$  has to be isomorphic to  $SL(4)$ ,  $Sp(4)$ ,  $SO(4)$  or  $SL(2) \times SL(2)$ . Note that (b) rules out  $SL(4)$ ,  $SO(4)$  and  $SL(2) \times SL(2)$ , and (c) rules out  $SL(2)$ . Thus only  $Sp(4)$  remains, and hence the proposition.

□

**Remark:** Our goal in a sequel will be to first construct modular Galois representations satisfying the Assumptions (1), (2) of Theorem 6.1 in the case  $B_2$ , which we think is possible over  $\mathbb{Q}$  and over function fields, by starting with cusp forms with suitable discrete series components, and then specialize to the function field case where we can appeal to the deep work of Lafforgue ([Laf]) on  $GL(n)$  and also use the recent striking results on the automorphic transfer to  $GL(n)$  from classical groups of Cogdell, Kim, Piatetski-Shapiro and Shahidi ([CoKPSS]), and also the backward lifting (automorphic descent) of Ginzburg, Rallis and Soudry (for the survey, see [S2005]).

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